

Rotation of the leaky dielectric particle in a rotating electric field

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We study the rotation of a weakly conducting particle around its axis of symmetry under the action of the external electric field, which spins in the plane normal to the axis of symmetry of the particle. The particle is embedded in a homogeneous stationary medium with finite electric conductivity permittivity that are different from the corresponding parameters of the particle. We determined the dependence of the particle angular velocity upon the amplitude angular velocity of the electric field. It is shown that depending upon the ratios of the particle electric conductivity permittivity to the corresponding parameters of the host medium the direction of rotation of the particle can be identical or opposite to the direction of rotation of the external electric field. We determined the amplitude-dependent critical angular velocity of the external electric field that separates the domains with two possible regimes of rotation of the particle. In the first domain the particle rotates only in one direction, while in the second domain the particle may rotate in two directions. We investigated also the stability of different regimes of rotation of the particle.

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I. INTRODUCTION

The behavior of small particles embedded in a host medium under the action of an external electric field has been the subject of numerous studies (see, e.g., Refs. [1,2], and references therein). In a case of the ideal dielectric the mathematical formulation of this problem is well known [3,4]. Taking into account the effects of finite electric conductivity requires using the approach that differs from that employed in the case of the ideal dielectric. For a hydrodynamic system such an approach is based on the system of equations of electrohydrodynamics [5,6].

The principal physical difference between a system with a finite electric conductivity and the ideal dielectric system is in the appearance of the additional free charge. In contrast to the polarization charge, which is formed during small (microscopic) times, this additional free charge is formed during macroscopic relaxation time. When the electric conductivity is low the characteristic relaxation time of the free charge is of the same order of magnitude as the macroscopic time scale of mechanical motions in the system. This additional free charge is localized at the boundary between the particle and a host medium. For finite relaxation time of the free charge the direction of the dipole moment of the particle \vec{P} may not coincide with the direction of the external electric field \vec{E} . This results in nonzero torque acting at the particle $\vec{M} = \vec{P} \times \vec{E}$ and the appearance of a number of effects. Thus, e.g., studies [7–10] showed that under the action of the alternating external electric field the final orientation of a spheroidal particle may be normal to the orientation of the ideal dielectric particle subjected to the same electric field. The comprehensive analysis of this effect was presented in Refs. [9,10].

Another effect, which is caused by the finite electric conductivity and is considered in numerous studies (see, e.g., Refs. [2,11–15]), is particle rotation under the action of a constant electric field. There exists a stationary regime whereby a particle rotates with a constant angular velocity around its axis of symmetry while a dipole moment of a particle \vec{P} and electric field \vec{E} remain constant. This stationary rotation is determined by a balance between the torque caused by viscous forces exerted on the particle by the surrounding fluid and torque \vec{M} due to the external electric field and also by the balance between the ohmic and convective electric currents that provides a constant direction of the dipole moment \vec{P} . These balances can be sustained only under the following conditions (see, e.g., Ref. [1]):

$$\frac{\sigma_1}{\sigma_2} > \frac{\varepsilon_1}{\varepsilon_2}, \quad E^2 > E_c^2, \quad (1)$$

where σ_1 and σ_2 are conductivities of the host medium and a particle, respectively, ε_1 and ε_2 are permittivities, E_c is a critical magnitude of the electric field that depends upon the viscosity of a host medium, geometry of the particle, and parameters $\varepsilon_1/\varepsilon_2$ and σ_1/σ_2 . If one of these two conditions is not satisfied then a particle embedded in a stationary fluid will remain at rest. Some additional discussions of this effect can be found in Refs. [2,12].

The situation is different if the direction of the external electric field changes with time. In the ideal dielectric a dipole moment is tuned instantaneously to the direction of the electric field. When the direction of the external electric field is normal to the axis of symmetry then the particle remains at rest since the dipole moment \vec{P} in this case is always aligned with the external electric field. In the nonideal dielectric, due to the torque caused by a free charge, the rotating external field (captures) the particle although the particle does not rotate synchronously with the electric field. Depending on its parameters the particle can rotate in the direction of rotation

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of the external field or in the opposite direction. In this sense a particle may have a positive electroviscosity (PEV) particle or a negative electroviscosity (NEV) particle. Electrodynamical parameters of NEV particles, ε_1, σ_1 , satisfy the condition (1), and in the case of the electric field with a constant direction NEV particles may rotate if the amplitude of the external electric field $E^2 > E_c^2$. At the same time PEV particles do not satisfy the conditions (1) and they may rotate only under the action of the rotating electric field. It is shown below that in the case of a rotating electric field there exists the critical threshold $E_c(\nu)$ (ν is a rotation frequency of the external electric field) such that when the amplitude of the electric field $E > E_c(\nu)$ the NEV particles may rotate either in the direction of the rotation of the external field or in the opposite direction, while PEV particles always rotate in the direction of the rotation of the external electric field independent of its amplitude. When the amplitude of the external electric field $E < E_c(\nu)$ there exists only one regime of rotation for PEV and NEV particles.

The goal of this study is to determine the dependence of the angular velocity of particle rotation on the amplitude and rotation frequency of the external electric field in the whole range of the parameters of the problem and to investigate the stability of different regimes of particle rotation.

In Sec. II we consider the mathematical model, which is used to derive the algebraic equations that determine the stationary solutions. These solutions describe particle rotation with a constant angular velocity ω_0 and rotation of the dipole moment \vec{P} with the frequency of the external electric field ν such that the angle between the vectors \vec{P} and \vec{E}_0 remains constant. In Sec. III we present the exact solutions of these equations and analyze the dependence of the frequency of particle rotation upon various parameters of the problem. We determined formulas for the critical frequency of rotation of the external field ν_c as a function of the field amplitude. It is shown that for $\nu < \nu_c$ there are two stable regimes of particle rotation and one unstable regime. When $\nu > \nu_c$ there exists only one stable regime of rotation. In Sec. IV we investigate the stability of particle rotation regimes as a function of the parameters of the problem taking into account the inertia of the particle. In the Conclusions we summarize the qualitative behavior of the system in various ranges of parameters. We discuss the similarity between a rotation of a particle immersed in a stationary fluid under the action of the rotating electric field and the dynamics of a particle in a fluid flow with shear under the action of the unidirectional external electric field.

It must be noticed that in the majority of studies it is implied that either the axis of the particle's rotation is fixed and is directed perpendicular to the external field or the particle has a spherical shape. The more complicated case when the axis of rotation of the asymmetric particle is not fixed was investigated in Refs. [11,13,14] and was not considered in this study.

II. MATHEMATICAL MODEL

Consider an axially symmetric particle with permittivity ε_2 and conductivity σ_2 embedded in a host medium with

permittivity ε_1 and conductivity σ_1 . Assume that the whole system is subjected to the electric field

$$\vec{E} = E_0(t)[\cos(\beta(t))\vec{e}_1 + \sin(\beta(t))\vec{e}_2]. \quad (2)$$

This electric field can be produced by the superposition of two perpendicular electric fields, $\vec{E}_1(t) = E_1(t)\vec{e}_1$ and $\vec{E}_2(t) = E_2(t)\vec{e}_2$. Clearly, $E_0(t) = \sqrt{E_1^2(t) + E_2^2(t)}$ and $\text{ctg}(\beta(t)) = E_1(t)/E_2(t)$. The electric field \vec{E} is normal to the axis of symmetry of the particle, and the angular velocity of the particle is directed along the axis of symmetry, $\vec{\Omega}(t) = \vec{e}_3\Omega(t)$, $\vec{e}_3 = \vec{e}_1 \times \vec{e}_2$.

The dynamics of a rotating particle is determined by the following equation:

$$I \frac{d\vec{\Omega}}{dt} = -\xi\vec{\Omega} + \vec{P} \times \vec{E}, \quad (3)$$

where I is the moment of inertia with respect to the axis of the particle, ξ is the rotational friction coefficient that is related with the viscosity coefficient η , $\xi = f_\xi \eta V$, f_ξ is the numerical coefficient that depends upon the shape of a particle, V is the particle volume, \vec{E} is the electric field applied to the system and determined by Eq. (2).

Hereafter we consider a spheroidal particle with the axis of symmetry \vec{e}_3 that coincides with the fixed axis of rotation. Therefore the axis of rotation of the particle cannot change its direction. The external electric field \vec{E}_0 is directed perpendicular to the axis of symmetry of the particle. In this case the principal axes of the ellipsoid \vec{e}_1, \vec{e}_2 can be chosen arbitrary in the plane perpendicular to the axis of symmetry, \vec{e}_3 , and depolarization factors $n_1 = n_2$.

The dipole moment of the particle \vec{P} can be written as $\vec{P} = \vec{P}_e + \vec{P}_\sigma$. Here \vec{P}_e is the dipole moment of the ideal dielectric particle that coincides with \vec{P}^∞ in Ref. [13]. In the problem considered in the present study $\vec{P}_e = \varepsilon_0 \varepsilon_1 \Gamma_e V \vec{E}(t)$, where Γ_e is the geometric coefficient. In the case when the electric field is directed perpendicular to the axis of symmetry of the particle and the particle has the shape of a spheroid, $\Gamma_e = \kappa_\varepsilon / (1 + \kappa_\varepsilon(1-n)/2)$, $\kappa_\varepsilon = \varepsilon_2/\varepsilon_1 - 1$, n is the depolarization factor along the axis of symmetry. In a case of a spherical particle $n = 1/3$, $n = 0$ for a cylindrical particle, and $n = 1$ for a disc. Inspection of the formula for \vec{P}_e shows that this component of the total dipole moment does not contribute to the torque acting on the particle $\vec{M} = \vec{P} \times \vec{E}$. The retarded component of the total dipole moment $\vec{P}_\sigma = \vec{P} - \vec{P}_e$ is determined by macroscopic relaxation equation (see Refs. [11–13]), which is written below without a subscript σ .

$$\frac{\partial \vec{P}}{\partial t} - \vec{\Omega} \times \vec{P} + \frac{\vec{P}}{\tau_m} = \frac{\kappa_\sigma - \kappa_\varepsilon}{(1 + f_\sigma)(1 + f_\varepsilon)} \frac{\vec{P}_s}{\tau_m}, \quad (4)$$

where $\tau_m = \tau_0(1 + f_\varepsilon)/(1 + f_\sigma)$, $\vec{P}_s = \varepsilon_0 \varepsilon_1 V \vec{E}(t)$, $f_\varepsilon = \kappa_\varepsilon(1-n)/2$, $f_\sigma = \kappa_\sigma(1-n)/2$, $\kappa_\sigma = \sigma_2/\sigma_1 - 1$, $\tau_0 = \varepsilon_0 \varepsilon_1 / \sigma_1$, and Ω is the particle rotation frequency.

It must be emphasized that validity to neglect \vec{P}_e hereafter follows not only from a condition that \vec{P}_e is directed along the field and consequently does not contribute to the torque \vec{M} , but also from the implied assumption that the rotation axis is fixed and cannot change its position in space. In the opposite case the sufficiently large electric field can destabilize the orientation of the particle [11,13,14]. For dielectric suspensions where the axes of particles rotation can change their direction this effect is negligible only in the case of slightly asymmetric nearly spherical particles.

Before solving the problem, i.e., determining stationary regimes of particle rotation in a spinning electric field, let us introduce dimensionless variables. The strength of the electric field \vec{E} is measured in units of $\vec{E} = \sqrt{\xi/\tau_m} V$. Consequently, the dimensionless electric field $\vec{X} = \vec{E}/\vec{E}$, and the dimensionless dipole moment can be written as $\vec{\Pi} = \vec{P}/\vec{E}V$. The dimensionless frequency of particle rotation $\omega = \Omega\tau_m$, and the dimensionless angular velocity of the external electric field can be written as $\nu = \partial\beta/\partial\tau$, where $\tau = t/\tau_m$ and the angle β is determined by Eq. (2).

Expression (2) yields the following formula for the dimensionless electric field \vec{X} :

$$\vec{X}(\tau) = \frac{X_0(\tau)}{2} \sum_{\alpha=\pm 1} e^{-i\alpha\beta(\tau)} \vec{u}_\alpha, \quad (5)$$

where $\vec{u}_\alpha = \vec{e}_1 + i\alpha\vec{e}_2$, $\alpha = \pm 1$.

Similarly, $\vec{\Pi}(\tau)$ can be written as follows:

$$\vec{\Pi}(\tau) = \sum_{\alpha=\pm 1} \Pi_\alpha(\tau) e^{-i\alpha\beta(\tau)} \vec{u}_\alpha. \quad (6)$$

Then the system of equations (3) and (4) can be rewritten as follows:

$$K \frac{d\omega}{d\tau} + \omega = -iX_0(\tau) \sum_{\alpha=\pm 1} \alpha \Pi_\alpha(\tau), \quad K = \frac{I}{\xi\tau_m}, \quad (7)$$

$$\left[\frac{\partial}{\partial\tau} + i\alpha(\omega - \nu) + 1 \right] \Pi_\alpha = -\frac{\chi X_0(\tau)}{2}, \quad \chi = \frac{\varepsilon_0 \varepsilon_1 (\kappa_\varepsilon - \kappa_\sigma)}{(1 + f_\varepsilon)(1 + f_\sigma)}. \quad (8)$$

The analysis performed in this study shows that the case when parameter $\chi > 0$ corresponds to a negative electroviscosity (NEV particles) while the case when $\chi < 0$ corresponds to a positive electroviscosity (PEV particles). Although in this connection it seems logical to replace parameter $\chi \rightarrow -\chi$, formally it is more convenient to determine parameter χ as in Eqs. (8).

Consider a case when angular velocity and amplitude X_0 of the external electric field are time independent. Then the system of equations (7) and (8) admits a stationary solution, $\Pi_\alpha = \Pi_\alpha^0$, $\omega = \omega_0$.

$$\Pi_\alpha^0 = -\frac{1 - i\alpha(\omega_0 - \nu)}{1 + (\omega_0 - \nu)^2} \frac{\chi X_0}{2}, \quad (9)$$

$$\omega_0 = \frac{\chi X_0^2 (\omega_0 - \nu)}{1 + (\omega_0 - \nu)^2}. \quad (10)$$

Equations (6) and (9) determine a polarization vector for the external field spinning with a constant angular velocity $[\beta(\tau) = \nu\tau]$, and Eq. (10) determines the angular velocity of a rotating particle. Prior to solving Eq. (10) let us investigate this equation in the limiting cases, $\omega_0 \gg \nu$ and $\omega_0 \ll \nu$. Assume that $\omega_0 \ll \nu$. Then Eq. (10) yields

$$\omega_0 = -\frac{\chi X_0^2 \nu}{1 + \nu^2}. \quad (11)$$

Equation (11) shows that a condition that $\omega_0 \ll \nu$ is valid for any values of ν , provided that $\chi X_0^2 \ll 1$. Therefore the latter condition is sufficient for the validity of Eq. (11) independent on ν . Note that a particle with $\chi > 0$ rotates in the direction opposite to the direction of spinning of the external field.

The frequency of particle rotation ω_0 can be larger than the rotation frequency of the external electric field ν , $|\omega_0| > |\nu|$, only when $\chi > 0$. When $\chi < 0$, Eq. (10) implies that $|\omega_0| < |\nu|$. It is shown below that in the case $\chi > 0$ the particle can rotate either in the direction of the rotation of the external electric field or in the opposite direction.

Now let us determine the solutions of Eq. (10) and investigate their stability. Denote $\omega_- = \omega_0 - \nu$ and rewrite Eq. (10) with respect to ω_- ,

$$\omega_-^3 + \nu\omega_-^2 + (1 - \chi X_0^2)\omega_- + \nu = 0. \quad (12)$$

Equation (12) implies that $\omega_- = 0$ only if $\nu = 0$, i.e., regimes with synchronous rotation of the particle and the vector of the external field do not exist. Equation (12) implies also that when $\chi X_0^2 \neq 0$ and $\nu \neq 0$ then $\omega_0 \neq 0$, i.e., the particle subjected to a rotating electric field cannot remain in rest as for $\chi > 0$ as in the case when $\chi < 0$.

III. ANALYSIS OF REGIMES OF PARTICLE ROTATION

In order to write the solutions of Eq. (12), introduce the following parameters:

$$a_1 = 9 \left(1 + \frac{\chi X_0^2}{2} \right), \quad a_2 = 3(1 - \chi X_0^2), \quad (13)$$

$$d_1 = \nu(\nu^2 + a_1^2), \quad d_2 = |a_2 - \nu^2|. \quad (14)$$

Using parameters (13) and (14) allows us to identify three domains. The first domain $\nu^2 < a_2$ is realized under the condition $\chi X_0^2 < 1$ only. In the opposite case when $\chi X_0^2 > 1$, $a_2 < 0$. In the first domain there exists only one real root $\omega_- = -\nu/3 + y_1$, where

$$y_1 = \frac{1}{3} \sum_{\alpha=\pm 1} \alpha (\sqrt{d_1^2 + d_2^3} - \alpha d_1)^{1/3}. \quad (15)$$

The second domain is determined by relations $\nu^2 > a_2$ and $|d_1| > \sqrt{d_2^3}$. There is also only one root in this domain, $\omega_- = -\nu/3 + y_2$, where

$$y_2 = -\frac{1}{3} \operatorname{sgn}(d_1) \sum_{\alpha=\pm 1} (|d_1| + \alpha \sqrt{d_1^2 - d_2^2})^{1/3}. \quad (16)$$

When $\nu^2 = a_2$ then $d_2 = 0$ and $y_1 = y_2$. The third domain with three real roots is determined by relations $\nu^2 > a_2$ and $|d_1| \leq \sqrt{d_2^3}$.

Denote real roots as $y_a < y_b < y_c$, where

$$y_a = -2 \cos(\varphi_0), \quad y_b = 2 \cos\left(\varphi_0 + \frac{\pi}{3}\right),$$

$$y_c = 2 \cos\left(\varphi_0 - \frac{\pi}{3}\right),$$

$$\varphi_0 = \cos^{-1}\left(\frac{d_1}{\sqrt{d_2^3}}\right). \quad (17)$$

The formula for the relative rotation frequency in this domain, ω_- , reads

$$\omega_-^i = -\frac{\nu}{3} + \frac{\sqrt{d_2}}{3} y_i, \quad (18)$$

where $i = a, b, c$.

Equation (12) implies that rotation frequency of the particle, $\omega_0 = \omega_- + \nu$, is an odd function of the frequency of the external electric field ν , $\omega_0(-\nu, \chi X_0^2) = -\omega_0(\nu, \chi X_0^2)$.

The above analysis allows us to identify two domains of the parameters of the problem. The first domain corresponds to the regimes of particle rotation that are determined by Eqs. (15) and (16). In this domain for a given value of parameter χ [see Eq. (8)] there exists only one regime of particle rotation. When the parameter $\chi < 0$, then the particle rotates in the direction of rotation of the external electric field. If $\chi > 0$ then the particle rotates against the direction of rotation of the external electric field. As we noted in Sec. II, particles with parameter $\chi < 0$ can be classified as particles with positive electric viscosity (PEV particles) while particles with $\chi > 0$ can be classified as particles with negative electric viscosity (NEV particles). The domain with a single rotation regime (single electroviscosity range) for a given amplitude of the external electric field X_0 is separated from the domain where the same particle can rotate in different directions when the parameter $\chi > 0$. In these bistable viscosity regions the frequencies of particle rotations are determined by Eqs. (17) and (18). For a given amplitude of the external electric field X_0 we can introduce the critical frequency $\nu_c(X_0)$ such that for the frequencies of rotation of the external electric field $|\nu| < \nu_c(X_0)$, the regime of particle rotation is determined by Eqs. (17) and (18) while for frequencies $|\nu| > \nu_c(X_0)$ the regime of particle rotation is determined by Eqs. (15) and (16). The above analysis implies that the dependence $\nu_c(X_0)$ is determined by the following conditions: $|d_1| = \sqrt{d_2^3}, \chi X_0^2 > 1$. These conditions yield the following relations:

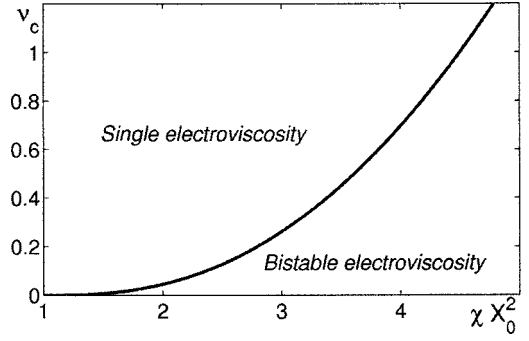


FIG. 1. Curve $\nu_c(\chi X_0^2)$ separates between the domains with single electroviscosity (SEV) and bistable electroviscosity (BEV).

$$\nu_c^2 = s + \sqrt{s^2 + (\chi X_0^2 - 1)^3}, \quad s = \frac{(\chi X_0^2)^2 - 20\chi X_0^2 - 8}{8}. \quad (19)$$

For $\chi X_0^2 \rightarrow 1$, $\nu_c^2 = 4/27(\chi X_0^2 - 1)^3$. In Fig. 1 we showed the dependence $\nu_c(\chi X_0^2)$. Equation (19) allows us to determine the critical magnitude of the amplitude of the external electric field $X_c(\nu)$ for a given frequency of rotation of the external field by substituting $\nu_c \rightarrow \nu$ and $X_0 \rightarrow X_c(\nu)$. For small amplitudes of the electric field $\chi X_0^2 - 1 \ll 1$, the formula for the critical amplitude reads

$$\chi X_c^2 = 1 + \frac{3}{4^{1/3}} |\nu|^{2/3}. \quad (20)$$

The bistable electroviscosity regime occurs only in the range $X_0^2 > X_c^2$. It is convenient to consider separately the cases with $\chi X_0^2 < 1$ and $\chi X_0^2 > 1$ for determining the dependence of the particle rotation frequency on the parameters of the problem.

In Fig. 2 we showed the dependence $\omega_0(\nu, \chi X_0^2)$ as a function of the parameter ν for various values of $\chi X_0^2 < 1$. It was noted above that the rotation frequency of the particle ω_0 is an odd function of the parameter ν . Inspection of the plots presented in Fig. 2 shows that for $\chi X_0^2 \ll 1$ Eq. (11) describes fairly well the dependence of the particle rotation frequency on the parameters of the problem.

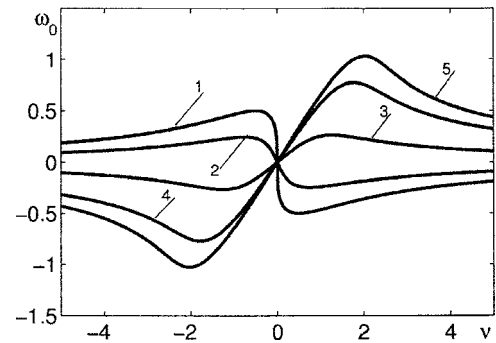


FIG. 2. Dependence of the rotation velocity of the particle ω_0 vs rotation frequency of the external electric field ν in the range of amplitude $\chi X_0^2 < 1$. 1: $\chi X_0^2 = 1$, 2: $\chi X_0^2 = 0.49$, 3: $\chi X_0^2 = -0.53$, 4: $\chi X_0^2 = -1.55$, 5: $\chi X_0^2 = -2.06$.

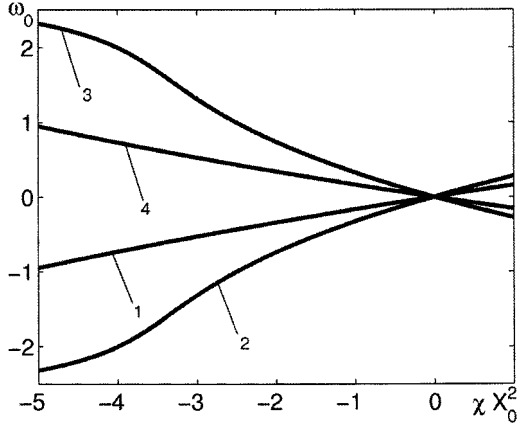


FIG. 3. Dependence of the particle rotation frequency ω_0 vs amplitude of the external electric field in the range of amplitude $\chi X_0^2 < 1$. 1: $\nu = -6$; 2: $\nu = -3$; 3: $\nu = 3$; 4: $\nu = 6$.

In Fig. 3 we showed the dependence $\omega_0(\nu, \chi X_0^2)$ as a function of the parameter χX_0^2 in the range of the amplitude of the external field $\chi X_0^2 < 1$.

The dependence of the particle rotation frequency in the range of the amplitude of the external field $\chi X_0^2 > 1$ is shown in Figs. 4 and 5.

In Fig. 4 we showed the dependence of the solutions $\omega_a < \omega_b < \omega_c$ vs the rotation frequency of the external electric field for different values of the amplitude of the field. Inspection of these plots demonstrates that $\omega_a(-\nu) = -\omega_c(\nu)$. In the bifurcation point $\nu = -\nu_c(\chi X_0^2)$ rotation regimes *a* and *b* merge, while rotation regimes *b* and *c* merge at $\nu = \nu_c(\chi X_0^2)$. The solution *b* exists only in the domain $|\nu| < \nu_c$ and when the amplitude of the external field $\chi X_0^2 > 1$. The equilibrium state of the particle under the action of the dc electric field for $\chi X_0^2 < 1$ is stable (see below and also Ref. [11]). This state of rest corresponds to the solutions *a* and *c* that merge for $\chi X_0^2 = 1$ when rotation frequency $\nu = 0$. For small frequencies of the external electric field when $|\nu| \ll \chi X_0^2 - 1$, the frequencies of particle rotation ω_a and ω_c are determined by formula (12), $\omega_c = \omega_0^+$, $\omega_a = \omega_0^-$, and $\omega_0^+(\nu) = -\omega_0^-(\nu)$.

In Fig. 5 we showed the dependence of the angular velocities of the particle $\omega_a < \omega_b < \omega_c$ vs the amplitude of the external electric field with different frequencies of rotation. Bifurcation amplitude X_c^2 is determined by Eq. (20) in the

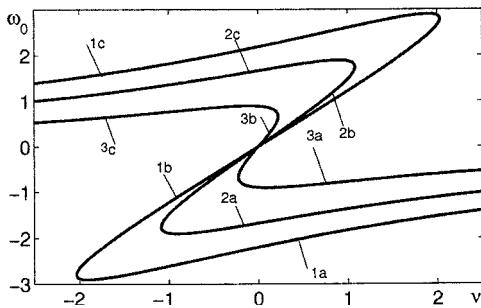


FIG. 4. Dependence of the particle rotation frequency ω_0 vs rotation frequency of the external electric field ν in the range $\chi X_0^2 > 1$ for three rotation regimes, $\omega_0^a, \omega_0^b, \omega_0^c$. 1a, 1b, 1c: $\chi X_0^2 = 5.8$; 2a, 2b, 2c: $\chi X_0^2 = 3.8$; 3a, 3b, 3c: $\chi X_0^2 = 1.8$.

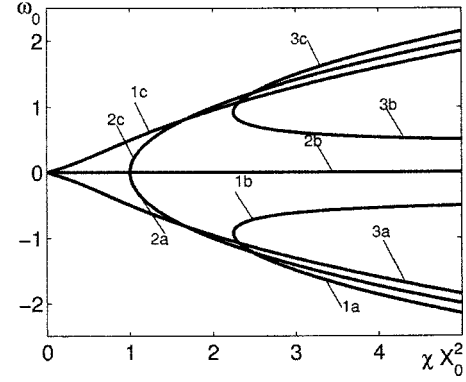


FIG. 5. Dependence of the particle rotation frequency ω_0 vs amplitude of the external electric field in the range $\chi X_0^2 > 1$ for three rotation regimes, $\omega_0^a, \omega_0^b, \omega_0^c$. 1a, 1b, 1c: $\nu = -0.4$; 2a, 2b, 2c: $\nu = 0$; 3a, 3b, 3c: $\nu = 0.4$.

range of small frequencies $\nu \ll 1$ and grows with the increase of the rotation frequency of the field.

The main regime of particle rotation remains continuous during transition through the bifurcation point $X_c(\nu)$. For $\nu > 0$ this regime corresponds to the solution *a*, and for $\nu < 0$ it corresponds to the solution *c*. In the main regime a particle with $\chi > 0$ rotates against the direction of rotation of the external electric field (NEV particle) while in additional regimes that appear in the bistable regime, a NEV particle behaves as a PEV particle, i.e., NEV particles rotate in the direction of rotation of the electric field.

IV. STABILITY OF PARTICLE ROTATION REGIMES

In the analysis of stability of the above determined regimes of stationary rotation of the particle we use Eqs. (7) and (8) and relations (9) and (10). Equations (7) and (8) are linearized in the vicinity of the stationary solutions that are determined by formulas (9) and (10). Consequently, we seek for the solutions of Eqs. (7) and (8), Π_α , in the form $\Pi_\alpha = \Pi_\alpha^0 + e^{\gamma t} \Pi_\alpha^1$ and $\omega = \omega_0 + e^{\gamma t} \omega^1$. Substituting these solutions to Eqs. (7) and (8) we obtain the system of linear homogeneous equations with respect to perturbations Π_α^1 and ω^1 ,

$$(K\gamma + 1)\omega^1 = -iX_0 \sum_{\alpha=\pm 1} \alpha \Pi_\alpha^1, \quad (21)$$

$$(\gamma + i\alpha\omega_- + 1)\Pi_\alpha^1 = -i\alpha\omega^1 \Pi_\alpha^0. \quad (22)$$

The condition for existence of the nontrivial solution to Eqs. (21) and (22) yields a dispersion equation with respect to the increment γ ,

$$\gamma^3 + a_1\gamma^2 + a_2\gamma + a_3 = 0, \quad (23)$$

where $a_1 = 2 + p$, $a_2 = 1 + \omega_-^2 + 2p - \chi X_0^2 p / (1 + \omega_-^2)$, $a_3 = p + p\omega_-^2 - p\chi X_0^2 (1 - \omega_-^2) / (1 + \omega_-^2)$, $p = 1/K = \xi \tau_m / I$. The condition for existence of the stationary stable rotation $\omega = \omega_0$ is $\text{Re}(\gamma) < 0$, and according to the Routh-Hurwitz (see, e.g., Ref. [18]) condition it realizes if and only if

$$a_1 a_2 > a_3, \quad a_3 > 0. \quad (24)$$

Using parameter $z=1+\omega_-^2$ the first condition in inequalities (24) can be rewritten as

$$\Delta_1(z) = z^2 + p \left(2 + p - \frac{\chi X_0^2}{2} \right) z - \frac{p^2 \chi X_0^2}{2} > 0. \quad (25)$$

The second condition in inequalities (24) can be written as

$$\Delta_2(z) = z^2 + \chi X_0^2 z - 2\chi X_0^2 > 0. \quad (26)$$

The stability condition is controlled by the parameters $\omega_-^2, \chi X_0^2, p$. Since $\omega_-^2(-\nu) = -\omega_-^2(\nu)$ (see Fig. 4) hereafter we consider only the domain $\nu > 0$. Since $z > 1$ it can be easily verified that when $\chi X_0^2 < 2$ the condition (25) is satisfied independently of other parameters. The condition (26) is satisfied independently of other parameters when $-2 < \chi X_0^2 < 1$.

Therefore, the stability of rotation of the PEV particle ($\chi < 0$) is determined by Eq. (26), and does not depend upon the inertia, i.e., on the parameter p . It can be shown that the rotation of PEV particles is always stable.

It was noted above that for $\chi > 0$ (NEV particle) there exist the main solution and additional solutions. The main solution is continuous during the transition through a bifurcation point, and describes particle rotation against the direction of rotation of the external electric field. The additional solutions occur in the bistable region and describe particle rotation in the direction of rotation of the external electric field. It must be emphasized that all the conclusions concerning the regimes a and c in the range $\nu > 0$ are valid for the regimes c and a , correspondingly, in the range $\nu < 0$.

When $\nu > 0$ the main regime is ω_0^a and the additional regimes are ω_0^b and ω_0^c . The regime ω_0^b is unstable in the whole range of parameters $p, \nu, \chi X_0^2 > 1$. Configurations of the stability domains for regimes ω_0^a and ω_0^c in the space of parameters $p, \nu, \chi X_0^2$ are different. We did not conduct the complete investigation of this facet of the problem. At this stage the performed numerical analysis of the stability of the main regime ω_0^a for $\nu > 0$ and the regime ω_0^c for $\nu < 0$ did not detect the domains of the instability. At the same time the additional regimes ω_0^c for $\nu > 0$ and ω_0^a for $\nu < 0$ revealed instability bands when parameter p was varied and parameters $\nu, \chi X_0^2$ were kept constant. Thus, e.g., there exist an interval of the parameter $p, p_1 < p < p_2$, where for given values of parameters $\nu > 0, \chi X_0^2$, solution ω_0^c is unstable. Similarly, in the same interval of parameter p the regime a is unstable for $\nu < 0$.

In the conclusion of this section let us consider the case $\nu=0$, which is useful for the analysis of a general case with $\nu \neq 0$ and that can be investigated completely.

In this case when $\chi X_0^2 < 1$, $\omega_- = \omega_0 = 0$, $z = 1$, $\Delta_1(z) > 0$, and $\Delta_2(z) > 0$. When $\chi X_0^2 > 1$ there exist three solutions, namely, $\omega_a = -\sqrt{\chi X_0^2 - 1}$, $\omega_b = 0$, $\omega_c = \sqrt{\chi X_0^2 - 1}$. Consequently, $z_a = z_c = \chi X_0^2$, $z_b = 1$, $\Delta_2^a = \Delta_2^c = 2\chi X_0^2(\chi X_0^2 - 1) > 0$, $\Delta_2^b = 1 - \chi X_0^2 < 0$. Thus the solution ω_b is unstable for $\nu=0$. It was noted above that this instability remains also for any $\nu \neq 0$. For $\nu=0$ the functions $\Delta_1^a = \Delta_1^c = \Delta_1$ and

$$\Delta_1 = \frac{\chi X_0^2}{2} [\chi X_0^2(2-p) + 4p + p^2]. \quad (27)$$

Taking into account that $\chi X_0^2 > 1$, for $p < 2$ Eq. (27) implies that independent of the amplitude of the external field the system is always stable. When $p > 2$ the condition $\Delta_1 > 0$ yields

$$\chi X_0^2 < \frac{p(4+p)}{p-2}. \quad (28)$$

Expression (28) corresponds to the condition of instability $\chi X_0^2 > p(4+p)/(p-2)$ that was derived in Ref. [12]. However, in study [12] it was not mentioned that for $p < 2$ the system is always stable and not vice versa as it would have been implied by the condition of instability that was derived in Ref. [12].

The minimum value of the right-hand side of Eq. (28) is attained for $p_* = 2 + \sqrt{12}$. Therefore when $1 < \chi X_0^2 < 2\sqrt{12} + 8$ then for $\nu=0$ the particle rotation is stable independent of the magnitude of the parameter p . The maximum particle rotation frequency in this case is $\omega_* = \sqrt{2\sqrt{12} + 7}$. When $\chi X_0^2 > 2\sqrt{12} + 8$ there exists a range of parameter $p_1 < p < p_2$, where the rotation of the particle is unstable. The end points of this interval are determined by the following formula:

$$p_{1,2} = \frac{\chi X_0^2 - 4}{2} \mp \frac{1}{2} \sqrt{(\chi X_0^2 - 4)^2 - 8\chi X_0^2}. \quad (29)$$

For $\chi X_0^2 \rightarrow \infty$, $p_1 \rightarrow 2, p_2 \rightarrow \infty$.

V. CONCLUSIONS

We showed that under the action of the homogeneous electric field rotating with a constant frequency ν the particle embedded in a host fluid with a finite electric conductivity rotates around its axis of symmetry when the external field is perpendicular to the axis. For a given amplitude of the external electric field X_0 and its frequency ν , the frequency of particle rotation ω_0 and the direction of rotation are determined by parameter χ [see Eq. (8)]. When $\chi < 0$ the particle rotates in the direction of rotation of the external field, and the frequency of particle rotation $|\omega_0| < |\nu|$. For small amplitudes of the external field the dependence $\omega_0(\nu)$ is determined by Eq. (11). In a more general case the dependence of particle rotation frequency on the frequency of rotation of the external electric field is shown in Fig. 2 (curves 3, 4, 5). The dependence of particle rotation frequency on the amplitude of the external electric field for small amplitudes is given by Eq. (11) while the dependence for a more general case is shown in Fig. 3.

For $\chi > 0$ the situation is more involved. In this domain for the given amplitude of the external field X_0 there exists the critical frequency $\nu_c(X_0)$ such that for $|\nu| < \nu_c(X_0)$ (see Fig. 1) there are three different regimes of particle rotation, $\omega_0^a < \omega_0^b < \omega_0^c$ (see Fig. 4 where $\nu_c = 0.4, 1, 2$).

In the main regime of rotation, which is continuous in the whole range of the parameters, and corresponds to the solution ω_0^a for $\nu > 0$, the particle rotates against the direction of rotation of the electric field. In the secondary regimes, ω_0^b

and ω_0^c for $\nu > 0$ or ω_0^a and ω_0^b for $\nu < 0$, the particle rotates in the direction of rotation of the external field, and the solution ω_0^b is unstable in the whole range of the parameters. The amplitude of the electric field that satisfies the condition $\chi X_0^2 = 1$ is a threshold. For the external electric field with $\chi X_0^2 < 1$ there exists only one regime of rotation of the particle independent of the rotation frequency of the external field.

The performed analysis elucidates some aspects of Quincke rotation (see, e.g., Ref. [2]). Since the dc electric field can be viewed as a particular case with $\nu = 0$, the obtained results show that the rest state of the particle, which can be realized for arbitrary amplitude of the field for $\chi < 0$, is not the stable state separated by the gap from the state of rotation, but it is rather the consequence of the fact that for $\chi < 0$, $|\omega_0| < |\nu|$, and $\omega_0 \rightarrow 0$ when $\nu \rightarrow 0$.

Although the main goal of this study is investigating the case of the rotating electric field it must be noted that in the case of a spherical particle this problem is formally equivalent to the case of rotation of the suspension particles in the shear flow in the presence of dc electric field [15–17]. Considering the latter case corresponds to the transformation to the reference frame that rotates with the external electric field. In this frame of reference the external electric field does not change its direction, the fluid rotates with the frequency $-\nu$ and the particle rotation frequency $\omega' = \omega - \nu = \omega_-$. Indeed, neglecting the moment of inertia in Eqs (7) and (8) (set $K = 0$) and performing the above substitution demonstrates the equivalence of the system of Eqs. (7) and (8) to the system of equations that was considered in Ref. [17] [see Eqs. (1) and (2) in Ref. [17]].

The results obtained in this study complement the results of previous investigations [13–17], which considered the renormalization of the viscosity of particle suspension in the presence of the dc electric field. The obtained results allow us to determine in the explicit form the critical shear $\Omega_c(X_0)$ that separates the domain of bistability from the domain where occurs only the main regime. The obtained results also allow us to determine the critical amplitude of the external electric field $X_c(\Omega)$ as a function of vorticity of the shear flow $\vec{\Omega} = 1/2\vec{\nabla} \times \vec{V}$ and the explicit dependence of the effective viscosity upon the amplitude of the electric field for various magnitudes of shear provided that the formula for the renormalization of the viscosity is known. Since the main interest in the latter problem is to compare the obtained results with the experimental data on the viscosity of suspensions subjected to the dc electric field, it will be considered in a separate study.

On the other hand, investigation of the viscosity of suspensions requires also solving the hydrodynamic part of the problem together with the analysis of the individual particle behavior. This problem does not have a universal solution, and the analysis of the adequacy of the available approaches in the literature requires their comparison with the experimental data. Therefore from the theoretical point of view it is preferable to investigate an individual rotator embedded in a host fluid under the action of the rotating electric field. In the case of the dc electric field this problem was considered in Ref. [12]. The obtained theoretical results allow us to perform a comparison between the experimental and theoretical results in two domains, $\chi > 0$ and $\chi < 0$.

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